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# Some Results on the $q$-Beta Function

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Abstract. In this article we find some results on the $q$-analogue of the beta function via using the concepts of neutrix and neutrix limit.

## 1. Introduction

The $q$-beta function $B_q(x,y)$ is defined for $x, y > 0$ by

$$B_q(x,y) = \int_0^1 t^{x-1}(1-qt)_q^{y-1} d_q t$$

see [1], and the integral diverging for $x, y \leq 0$.

By using the concepts of neutrix and neutrix limit see [2],the $q$-beta function $B_q(x,y)$ is defined for all values of $x$ and $y$, see [3] and its derivative by first parameter is defined as

$$B_q^{(r,0)}(x,y) = \mathrm{N}\!-\!\lim_{\varepsilon\to 0} \int_\varepsilon^{1-\varepsilon} t^{x-1} \ln^r t (1-qt)_q^{y-1} \tag{1}$$

for $x, y \neq 0, -1, -2, \dots$ and $r \in \mathbb{N}$, see [4].
In the following, we let $\mathcal{N}$ be the neutrix having domain $N' = \{\varepsilon : 0 < \varepsilon < \infty\}$ and range $N''$ the real numbers, with the negligible functions being finite linear sums of the functions

$$\varepsilon^\lambda \ln^{r-1}, \ln^r \varepsilon, \quad (\lambda < 0, r \in \mathbb{N})$$

and all functions of $\varepsilon$ which converge to zero in the normal sense as $\varepsilon$ tends to zero.

One of the differences between quantum and ordinary calculus is that the derivative of the product of two functions is not symmetric. Because of this, the $q$-integration by parts can be given in two different ways. First one was given in [1] and the second one which we will use throughout in the paper, is that

$$\int_a^b f(qx) d_q g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x) d_q f(x). \tag{2}$$

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In this work we find some results for the derivative of the $q$-beta function.

## 2. Main Results

Before we obtain main results, we need the following lemma which can be shown easily by induction.

**Lemma 2.1.** *We have for* $r = 0, 1, 2, \ldots$

$$\mathrm{N}-\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln^r t d_q t = 0$$

*and the neutrix limit of the integral*

$$\int_{\varepsilon}^{1-\varepsilon} t^{j}\ln^r t d_q t$$

*exists as* $\varepsilon \to 0$ *for* $j \neq -1$ *and* $r = 0, 1, 2, \ldots$.

**Theorem 2.1.** *For* $r, n = 1, 2, \ldots$ *we have*

$$B_q^{(r,0)}(0,1) = 0 \tag{3}$$

$$B_q^{(r,0)}(0,0) = B_q^{(r,0)}(1,0) \tag{4}$$

$$\begin{aligned} B_q^{(r,0)}(0,n+1) = &-(q^n-1)\sum_{j=0}^{r}\binom{r}{j}\ln^{r-j-1} q^{-1} B_q^{j+1,0}(1,n) \\ &-\sum_{j=0}^{r-1}\binom{r}{j}\ln^{r-j-1} q^{-1} B_q^{j+1,0}(0,n+1) \end{aligned} \tag{5}$$

$$B_q^{(r,0)}(-1,0) = B_q^{(r,0)}(-1,1) + B_q^{(r,0)}(1,0) \tag{6}$$

(7)

$$B_q^{(r,0)}(-n-1,1) = \frac{\ln^r q^{-1}}{[-n-1]} + \frac{1}{q^{-n-1}-1}\sum_{j=0}^{r-1}\binom{r}{j}\ln^{r-j} q^{-1} B_q^{(j,0)}(-n-1,1)$$

*Proof.* It follows from the definition of $q$-derivative that

$$\int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln t d_q t = \frac{1}{2}\frac{q-1}{\ln q}\ln(1-\varepsilon)\ln q^{-1}(1-\varepsilon) - \ln\varepsilon\ln q^{-1}\varepsilon,$$

and

$$\mathrm{N}-\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln t d_q t = 0,$$

so equation (3) follows.

Now for (4), we have

$$B_q^{(r,0)}(0,0) = \mathrm{N}-\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln^r t(1-qt)_q^{-1} d_q t$$

$$
\begin{aligned}
&= \mathrm{N-}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln^r t\frac{1}{1-t}d_qt\\
&= \mathrm{N-}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln^r t d_qt + \mathrm{N-}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon}\frac{\ln^r t}{1-t}d_qt\\
&= B_q^{(r,0)}(1,0)
\end{aligned}
$$

for $r = 1, 2, 3, \ldots$.

For this time, let us take $f(qt) = \ln^r t(1-qt)_q^n$ and $g(t) = \frac{q-1}{\ln q}\ln t$. Then using $q$-integration by parts, we obtain

$$
\begin{aligned}
&\int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln^r t(1-qt)_q^n d_qt\\
&= \mathrm{N-}\lim_{\varepsilon\to 0}\frac{q-1}{\ln q}\ln(1-\varepsilon)\ln^r(q^{-1}(1-\varepsilon))(1-(1-\varepsilon))_q^n\\
&\quad - \frac{q-1}{\ln q}\ln(\varepsilon)\ln^r(q^{-1}\varepsilon)(1-\varepsilon)_q^n\\
&\quad - \frac{q-1}{\ln q}\int_{\varepsilon}^{1-\varepsilon}\ln t\Bigg[-[n]\sum_{j=0}^{r}\binom{r}{j}\ln^{r-j}q^{-1}\ln^j t(1-qt)_q^{n-1}\\
&\quad - \frac{(1-qt)_q^n}{q-1}\sum_{j=0}^{r-1}\binom{r}{j}\ln^{r-j}q^{-1}t^{-1}\ln^j t\Bigg]d_qt\\
&= \mathrm{N-}\lim_{\varepsilon\to 0}\frac{q-1}{\ln q}\ln(1-\varepsilon)\ln^r(q^{-1}(1-\varepsilon))(1-(1-\varepsilon))_q^n\\
&\quad - \frac{q-1}{\ln q}\ln(\varepsilon)\ln^r(q^{-1}\varepsilon)(1-\varepsilon)_q^n\\
&\quad - (q^n-1)\sum_{j=0}^{r}\binom{r}{j}\ln^{r-j-1}q^{-1}\int_{\varepsilon}^{1-\varepsilon}\ln^{j+1}t(1-qt)_q^{n-1}d_qt\\
&\quad - \sum_{j=0}^{r-1}\binom{r}{j}\ln^{r-j-1}q^{-1}\int_{\varepsilon}^{1-\varepsilon}t^{-1}\ln^{j+1}t(1-qt)_q^n d_qt.
\end{aligned}
$$

Taking the neutrix limit of both sides gives equation (5).

Next

$$
\begin{aligned}
&B_q^{(r,0)}(-1,0) = \mathrm{N-}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} t^{-2}\ln^r t(1-qt)_q^{-1}d_qt\\
&= \mathrm{N-}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} t^{-2}\ln^r t\frac{1}{1-t}d_qt\\
&= \mathrm{N-}\lim_{\varepsilon\to 0}\left\{\int_{\varepsilon}^{1-\varepsilon} t^{-2}\ln^r t d_qt + \int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln^r t d_qt + \int_{\varepsilon}^{1-\varepsilon}\ln^r t\frac{1}{1-t}d_qt\right\},
\end{aligned}
$$

and since the neutrix limit of second integral is zero in the last equation, we obtain

$$B_q^{(r,0)}(-1,0) = B_q^{(r,0)}(-1,1) + B_q^{(r,0)}(1,0).$$

In particular, for $r = 1$ we have

$$B_q^{(1,0)}(-1,0) = \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} t^{-2}\ln t(1-qt)_q^{-1} d_qt$$

$$= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} t^{-2}\ln t\frac{1}{1-t} d_qt$$

$$= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\left\{\int_\varepsilon^{1-\varepsilon} t^{-2}\ln t d_qt + \int_\varepsilon^{1-\varepsilon} t^{-1}\ln t d_qt + \int_\varepsilon^{1-\varepsilon} \ln t\frac{1}{1-t} d_qt\right\}$$

$$= -\frac{q\ln q}{[1]^2(q-1)} + B_q^{(1,0)}(1,0).$$

To obtain equation (7), let us take $f(t)$ and $g(t)$ as $f(qt) = ln^r q^{-1}t$ and $g(t) = \dfrac{t^{-n-1}}{[-n-1]}$ respectively and then we get

$$B_q^{(r,0)}(-n-1,1) = \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} t^{-n-2}\ln^r q^{-1}t d_qt$$

$$= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\frac{(1-\varepsilon)^{-n-1}}{[-n-1]}\ln^r q^{-1}(1-\varepsilon) - \frac{\varepsilon^{-n-1}}{[-n-1]}\ln^r q^{-1}\varepsilon$$

$$+ \frac{1}{[-n-1](q-1)}\sum_{j=0}^{r-1}\binom{r}{j}\ln^{r-j} q^{-1}\int_\varepsilon^{1-\varepsilon} t^{-n-2}\ln^j t d_qt.$$

This completes the proof. □

Also it should be noted that by taking $f(t) = \ln^r t$ and $g(t) = \frac{t^{-n-1}}{[-n-1]}$ and using $q$-integration by parts in [1] we have

$$\int_\varepsilon^{1-\varepsilon} t^{-n-2}\ln^r q^{-1}t d_qt = \frac{(1-\varepsilon)^{-n-1}}{[-n-1]}\ln^r(1-\varepsilon) - \frac{\varepsilon^{-n-1}\ln^r\varepsilon}{[-n-1]}$$

$$- \frac{1}{[-n-1](q-1)}\sum_{j=0}^{r-1}\binom{r}{j}\ln^{r-j} q\int_\varepsilon^{1-\varepsilon}(qt)^{-n-1}t^{-1}\ln^j t d_qt,$$

and taking the neutrix limit on both sides we get

$$(8)\quad B_q^{(r,0)}(-n-1,1) = \frac{1}{q^{n+1}-1}\sum_{j=0}^{r-1}\binom{r}{j}\ln^{r-j} qB_q^{(j,0)}(-n-1,1).$$

If we use property of the $q$-analogue of $(1-t)^n$ on $B_q^{(r,0)}(0,n)$ we get

$$B_q^{(r,0)}(0,n) = \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_\varepsilon^{1} t^{-1}\ln^r t(1-qt)_q^{n-1} d_qt$$

$$= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} t^{-1}\ln^r t(1-qt)_q^{n-2}(1-q^{n-1}t)d_qt$$
$$= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} t^{-1}\ln^r t(1-qt)_q^{n-2}d_qt - q^{n-1}\int_{\varepsilon}^{1}\ln^r t d_qt$$

and so

$$B_q^{(r,0)}(0,n) = \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} t^{-1}\ln^r t d_qt - \sum_{j=1}^{n-1} q^j\int_{\varepsilon}^{1}\ln^r d_qt.$$

Hence by lemma 2.1 we get

$$B_q^{(r,0)}(0,n) = -\sum_{j=1}^{n-1} q^j B_q^{(r,0)}(1,1). \tag{9}$$

We can similarly obtain that

$$B_q^{(r,0)}(1,n) = B_q^{(r,0)}(1,1) - \sum_{j=1}^{n-1} q^j B_q^{(r,0)}(2,1). \tag{10}$$

Now we generalize equation (6) as follows.

**Theorem 2.2.**

$$B_q^{(r,0)}(-n,0) = B_q^{(r,0)}(1,0) + \sum_{j=2}^{n+1} B_q^{(r,0)}(-j+1,1) \tag{11}$$

*for* $r,n = 1,2,\ldots$.

*Proof.* If we apply $q$-integration by parts on the definition of beta function, then we get

$$B_q^{(r,0)}(-n,0) = \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} t^{-n-1}\ln^r t(1-qt)_q^{-1}d_qt$$
$$= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\left\{\int_{\varepsilon}^{1-\varepsilon}\ln^r t(1-qt)_q^{-1}d_qt + \int_{\varepsilon}^{1-\varepsilon} t^{-1}\ln^r t d_qt + \sum_{j=2}^{n+1}\int_{\varepsilon}^{1-\varepsilon} t^{-j}\ln^r t d_qt\right\}.$$

Hence by lemma 2.1, the proof is completed. □

In next theorem, the constants $c_{r,n}(m)$ of the expansion of the terms

$$(1-\varepsilon)^{n-1}\ln^r(1-\varepsilon) = \sum_{i=0}^{\infty} c_{r,n}(i)\varepsilon^i$$

can be presented as

$$c_{r,n}(m) = \begin{cases} 0, & m < r \\ (-1)^m, & m = r \\ n - m, & m = r + 1 \end{cases}$$

for $r, m = 1, 2, \ldots$.

**Theorem 2.3.** *We have*

$$\begin{aligned} B_q^{(r,0)}(n,-m) &= \frac{c_{r,n}(m)}{m} + \frac{[n-1]}{[-m]} B_q^{(n,0)}(n-1,-m+1) \\ &\quad + \frac{q^{n-1}}{[-m]} \sum_{k=0}^{r-1} \binom{r}{k} \ln^{r-k} q B_q^{(k,0)}(n-1,-m+1) \end{aligned} \tag{12}$$

*for* $r, m = 1, 2, \ldots$ *and* $n = 2, 3, \ldots$ .

*Proof.* Since we have

$$\begin{aligned} &\int_\varepsilon^{1-\varepsilon} t^{n-1} \ln^r t (1-qt)_q^{-m-1} d_q t \\ &= -\frac{t^{n-1} \ln^r t (1-t)_q^{-m}}{[-m]}\Big|_\varepsilon^{1-\varepsilon} - \frac{1}{[-m]} \Big\{ [n-1] \int_\varepsilon^{1-\varepsilon} t^{n-2} \ln^r t (1-qt)_q^{-m} d_q t \\ &\quad + \sum_{k=0}^{r-1} \binom{r}{k} \ln^{r-k} t \int_\varepsilon^{1-\varepsilon} (qt)^{n-1} \ln^k t (1-qt)_q^{-m} d_q t \Big\} \\ &= -\frac{(1-\varepsilon)^{n-1} \ln^r (1-\varepsilon) \varepsilon^{-m}}{[-m]} + \frac{\varepsilon^{n-1} \ln^r \varepsilon (1-\varepsilon)_q^{-m}}{[-m]} \\ &\quad + \frac{[n-1]}{[-m]} \int_\varepsilon^{1-\varepsilon} t^{n-2} \ln^r t (1-qt)_q^{-m} d_q t \\ &\quad + \frac{q^{n-1}}{[-m]} \sum_{k=0}^{r-1} \binom{r}{k} \ln^{r-k} t \int_\varepsilon^{1-\varepsilon} t^{n-1} \ln^k t (1-qt)_q^{-m} d_q t, \end{aligned}$$

the proof is completed by taking the neutrix limit of both side of the last equation. □

Finally, we note that all results obtained in the paper are in agreement with the results given in [5–10] as $q$ tends 1.

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